Boundary control and observation of some one-dimensional vibrating structures : regularity and stabilization

Juliette Leblond, Jean-Paul Marmorat

UNITÉ DE RECHERCHE
INRIA-LORRAINE

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

Rapports de Recherche

N° 1382

Programme 6
Calcul scientifique, Modélisation et Logiciels numériques

BOUNDARY CONTROL AND OBSERVATION OF SOME ONE-DIMENSIONAL VIBRATING STRUCTURES : REGULARITY AND STABILIZATION

Juliette LEBLOND
Jean-Paul MARMORAT

Janvier 1991

RR-1382

BOUNDARY CONTROL AND OBSERVATION OF SOME ONE DIMENSIONAL VIBRATING STRUCTURES: REGULARITY AND STABILIZATION

Juliette LEBLOND, Jean-Paul MARMORAT
Ecole des Mines de Paris, Centre de Mathématiques Appliquées,
Sophia-Antipolis, 06565 Valbonne, France

Abstract:
We study a class of second order evolution equations with unbounded input-output operators.
We establish sufficient conditions for the existence and regularity of both solution and output, and for observability.
This makes it possible to apply boundary output feedbacks which, in the linear case, lead to uniform exponential stability.
These hypotheses are shown to hold for a beam equation. They are only sufficient conditions: they do not hold for a wave equation which posseses however the same regularity and stability properties.

CONTROLE ET OBSERVATION FRONTIERE DE STRUCTURES VIBRANTES MONODIMENSIONNELLES: REGULARITE ET STABILISATION

Résumé:
On étudie une classe d'équations d'évolution du second ordre en temps, avec des opérateurs d'entrée-sortie non bornés.
On établit pour celles-ci des conditions suffisantes pour l'existence et la régularité de la solution et de la sortie, ainsi que pour l'observabilité.
Ces résultats légitiment l'application de feedbacks de sortie sur la frontière, fournissant dans le cas linéaire la stabilité exponentielle uniforme.
On montre que ces hypothèses sont vérifiées par une équation d'Euler-Bernoulli.
Ces conditions ne sont que suffisantes: elles ne sont pas toutes satisfaites par une équation des ondes qui possède cependant les mêmes propriétés de régularité et de stabilité.

Contents

1 Introduction

The aim of this work is to study the stabilization of second order (in time) evolution equations. Such equations may model the dynamical behaviour of some flexible structures.

We wish to synthesize closed-loop controls, physically realizable, which stabilize these structures. We consider unbounded input-output operators as models for sensors and actuators operating on a few parts or points of the boundary of spatial domain.

General results of Lions and Magenes[16] concerning the regularity of solutions for these open-loop systems would not hold when applying an unbounded observation operator to these solutions. On the other hand, Chen, Delfour, Krall, and Payre[4] proved that, under some hypotheses on the feedback operator, the closed-loop system is well-posed and admits regular solution and output.

We first establish regularity properties of both solution and output of the open-loop controlled system for an L^2-class of inputs. We then consider the effect of particular closed-loop controls. This makes it possible to compare the chosen input with others in the open-loop class, investigating the existence of an optimal control, the influence of delays and various perturbations, or robustness properties. We show that these closed-loop systems are observable, which leads to uniform exponential stability.

In §2 we recall some classical notations for variational formulation of partial differential equations, and introduce the Galerkin approximations of solutions. Moreover we present a result of Ball and Slemrod[1] on the behaviour of non-harmonic Fourier series.

In §3 we prove the existence and regularity of solution and output for a class of second order evolution equations in the case of unbounded input-output operators.
We also establish observability of these open-loop systems, from a characteristic time T_0.

In §4 we show that for a linear output feedback the system is still observable. This is the key point in the proof of the uniform exponential stability of such a closed-loop system, since the feedback depends only on the observation (and not on the whole state).

In §5 we apply these results to an Euler-Bernoulli beam clamped at one end, with point colocated sensor-actuator at the other end.

In §6 we consider a wave equation, which may model the torsion of a beam, with boundary control and observation. Although one of the hypotheses is not proved to be satisfied, we show that the conclusions of our main theorems remain true.

2 Statement of the problem

Let A be en unbounded, self-adjoint, coercive, linear operator on a Hilbert space H, with domain $\mathcal{D}_H(A)$. Let $V = \mathcal{D}(A^{1/2})$, with inner product $(v,w)_V = \left(A^{1/2}\,v, A^{1/2}\,w\right)_H$. We denote by the same symbol the unique operator $A \in \mathcal{L}(V,V')$ such that $(v,w)_V = \langle A\,v, w\rangle_{V'V}$.
We have

$$\mathcal{D}_H(A) \subset V \subset H \equiv H' \subset V'$$

with continuous and dense embeddings. We suppose that the operator $A^{-1} \in \mathcal{L}(H,H)$ is compact (this is true as soon as the embedding $V \subset H$ is compact).

Let U and Y be two finite dimensional spaces, and $T > 0$ be fixed. Call

$$\mathcal{U} = L^2(0,T;U)\,,\ \mathcal{Y} = L^2(0,T;Y)\,,$$

identifying $\mathcal{U}$ with $\mathcal{U}'$ and $\mathcal{Y}$ with $\mathcal{Y}'$.
Then, let $B \in \mathcal{L}(U,V')$, and $C \in \mathcal{L}(V,Y)$.

Consider the system

$$(2.1)\qquad \begin{cases} \ddot{y}(t) + A\,y(t) = B\,u(t) \\ z = C\,\dot{y} \\ y(0) = y_0,\ \dot{y}(0) = y_1 \end{cases}$$

for $y_0 \in V$, $y_1 \in H$, and $u \in \mathcal{U}$. Let $\{\lambda_k = \omega_k^2,\ \omega_k > 0\}_{k>0}$ and $\{\phi_k\}_{k>0}$ be the sequences of eigenvalues and eigenfunctions of A. $\{\phi_k\}_{k>0}$ is an orthogonal basis for V and H, with $\|\phi_k\|_V = \omega_k$ and $\|\phi_k\|_H = 1$, $\forall k > 0$. We set $\omega_{-k} = -\omega_k$, $\phi_{-k} = \phi_k$, $\forall k > 0$, and

$$\begin{aligned} b_k &= B^*\,\phi_k \in U \\ c_k &= C\,\phi_k \in Y \end{aligned}$$

System (2.1) is equivalent to

$$(2.2) \qquad \begin{cases} \dot{x}(t) = \mathcal{A}\, x(t) + \mathcal{B}\, u(t) \\ z = \mathcal{C}\, x \\ x(0) = x_0 \end{cases}$$

where

$$x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}, \quad x_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},$$

and

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 0 & C \end{pmatrix}.$$

The densely defined closed operator $\mathcal{A}$ generates a C_0-semigroup of contractions on $V \times H$. The domain of $\mathcal{A}$ is given by $\mathcal{D}(\mathcal{A}) = \mathcal{D}_H(A) \times V$.

The results in the following sections are established relatively to the first order system (2.2). We use the equivalent second order system (2.1) to carry out some demonstrations.

Call

$$\begin{aligned} X &= V \times H \\ \mathcal{X} &= L^\infty(0,T;X) \end{aligned}$$

Introduce

$$\begin{aligned} W_n &= \operatorname{span}\{\phi_k,\ 0 < k \leq n\} \\ W &= \operatorname{span}\{\phi_k,\ k > 0\} \end{aligned}$$

and

$$\begin{aligned} \overline{B}_n &= B^*|_{W_n} \\ B_n &= \overline{B}_n^* \end{aligned}$$

so that $B_n^*\, w = B^*\, \pi_n\, w$, for $w \in V$, where π_n is the projection from V to W_n. One can observe that

- W is dense in V, and $B_n^* \to B^*$ strongly, as $n \to \infty$, since $\{\phi_k\}_{k>0}$ is a basis of V

- $\mathcal{A}\, W_n \subset W_n$

We define

$$\mathcal{B}_n = \begin{pmatrix} 0 \\ B_n \end{pmatrix}$$

and Π_n as the projection of X on $W_n \times W_n$. System (2.2) is approximated by the finite dimensional system

$$\left\{ \begin{array}{l} \dot{x}_n(t) = \mathcal{A}\, x_n(t) + \mathcal{B}_n\, u(t) \\ x_n(0) = \Pi_n\, x_0 \\ z_n = \mathcal{C}\, x_n \end{array} \right. \tag{2.3}$$

We give now the form of the solutions $x_n(t)$ of (2.3). Let

$$S_k(t) = \begin{pmatrix} \cos \omega_k t & \dfrac{\sin \omega_k t}{\omega_k} \\ -\omega_k \sin \omega_k t & \cos \omega_k t \end{pmatrix}$$

For all sequences of real numbers $(\alpha_k)_{k>0}$, $(\beta_k)_{k>0}$, we have

$$\|\sum_{k=1}^{n} S_k(t) \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \phi_k\|_X^2 = \sum_{k=1}^{n} [\omega_k^2\, \alpha_k^2 + \beta_k^2] \tag{2.4}$$

Let

$$x_n(t) = x_n^a(t) + x_n^c(t),$$

where

$$x_n^a(t) = \sum_{k=1}^{n} S_k(t) \begin{pmatrix} (y_0, \phi_k)_H \\ (y_1, \phi_k)_H \end{pmatrix} \phi_k$$

is the autonomous part of $x_n(t)$, and

$$\begin{aligned} x_n^c(t) &= \sum_{k=1}^{n} \left[\int_0^t (u(\tau), b_k)_U \begin{pmatrix} \dfrac{\sin \omega_k (t-\tau)}{\omega_k} \\ \cos \omega_k (t-\tau) \end{pmatrix} d\tau \right] \phi_k \\ &= \sum_{k=1}^{n} \left[S_k(t) \int_0^t (u(\tau), b_k)_U \begin{pmatrix} -\dfrac{\sin \omega_k \tau}{\omega_k} \\ \cos \omega_k \tau \end{pmatrix} d\tau \right] \phi_k \end{aligned}$$

its controlled part. We shall also use the following equivalent expression for $x_n^c(t)$

$$(2.5) \qquad x_n^c(t) = \sum_{k=1}^{n} \left[(u(\tau), b_k)_U * \begin{pmatrix} \dfrac{\sin \omega_k \tau}{\omega_k} \\ \\ \cos \omega_k \tau \end{pmatrix} \right] (t)\, \phi_k$$

By construction $x_n(t)$ is the solution of (2.3) and we have

- $\lim_{n \to \infty} x_n(0) = x_0$ in X
- $\|x_n(0)\|_X \leq \|x_0\|_X$

We introduce the same decomposition for the output z_n of (2.3):

$$\begin{aligned} z_n &= z_n^a + z_n^c \\ z_n^a &= C\, x_n^a \\ z_n^c &= C\, x_n^c \end{aligned}$$

We also consider the family $(\hat{u}_k^t)_{k>0}$, defined for any $t \in [0, T]$ by

$$\hat{u}_k^t = \int_0^t u(\tau) e^{-i\omega_k \tau} d\tau$$

In the next sections we shall need the following result due to Ball and Slemrod[1] and Ingham[12]

Theorem 2.1 ([1]) *Let*

$$f(t) = \sum_{n=-\infty}^{+\infty} a_n\, e^{-i\omega_n t}$$

where the a_n are complex constants with only a finite number of them different from zero, and the ω_n are real and satisfy

$$\underline{\lim}_{n \to \infty}\, \omega_{n+1} - \omega_n \geq \alpha > 0\,.$$

Then

$$(2.6) \qquad \sum_n |a_n|^2 \geq \zeta \int_0^T |f(t)|^2\, dt,$$

where $\zeta > 0$ depends only on T and α.
If moreover

$$T > \frac{2\pi}{\alpha},$$

then

(2.7) $$\sum_n |a_n|^2 \le \eta \int_0^T |f(t)|^2\, dt,$$

where $\eta > 0$ depends only on T and α.

3 Open-loop systems

We consider here system (2.2) and establish existence, regularity, and observability properties for its solution and output, under some hypotheses related to

- spectral properties of operator A,
- behaviour of B and C,
- input-output mappings.

3.1 Existence and regularity

Theorem 3.1 gives sufficient conditions for the existence and regularity of a solution of (2.2) and of an associated output.

Theorem 3.1 *(i) Let A, B, and C be such that there exist some constants α, β, γ > 0 satisfying*

(3.1) $$\underline{\lim}_{k\to\infty} \omega_{k+1} - \omega_k \ge \alpha$$

(3.2) $$\left\{ \begin{array}{lcl} \|c_k\|_Y & \le & \gamma \\ \|b_k\|_U & \le & \beta \end{array} \right. , \forall k > 0$$

Then, for all $u \in \mathcal{U}$ and $x_0 \in X$, the system (2.2) has a unique solution $x \in C^0([0,T]; X) \subset \mathcal{X}$ such that the mapping

$$\begin{array}{rccc} \Lambda_x : & X \times \mathcal{U} & \longrightarrow & \mathcal{X} \\ & (x_0, u) & \longmapsto & x \end{array}$$

is continuous.

(ii) Moreover, if the input-output maps of the approximated system (2.3)

(3.3) $\{u \mapsto z_n^c\}_{n>0}$ *are equicontinuous from* $\mathcal{U}$ *into* $\mathcal{Y}$,

then the associated output $z = C\,x$ *is well-defined in* $\mathcal{Y}$ *and the mapping*

$$\begin{array}{rccc} \Lambda_z : & X \times \mathcal{U} & \longrightarrow & \mathcal{Y} \\ & (x_0, u) & \longmapsto & z \end{array}$$

is continuous.

Proof of Theorem 3.1

Part (i)

We first establish an inequality involving the solutions $x_n(t)$ of the finite dimensional system (2.3), using Theorem 2.1. Then we apply existence theorems due to Lions and Magenes in [16], and the convergence properties of $x_n(t)$ when n goes to infinity.
To do this we need the following lemma. We note $l^2(U) = l^2(I\!N, U)$.

Lemma 3.1 *Under hypothesis* (3.1), *there exists a constant* $c > 0$ *such that*

$$u \in \mathcal{U} \Longrightarrow \left\{ \begin{array}{l} (\hat{u}_k^t)_{k>0} \in l^2(U) \\ \\ \|(\hat{u}_k^t)_{k>0}\|_{l^2(U)} \leq c\,\|u\|_{\mathcal{U}} \end{array} \right. \qquad \forall t \in [0, T]$$

Proof of Lemma 3.1

Consider $(v_k)_{k>0} \in l^2(U)$ and $v \in \mathcal{U}$ defined by

$$v(\tau) = \sum_{k=1}^{n} v_k\, e^{i\omega_k \tau}.$$

Inequality (2.6) of Theorem 2.1 gives

$$\|v\|_{\mathcal{U}} \leq \frac{1}{\sqrt{\zeta}}\, \|(v_k)_{k>0}\|_{l^2(U)}\,.$$

Moreover, for all $u \in \mathcal{U}$, for all $t \in [0, T]$

$$\begin{aligned}(u,v)_{L^2(0,t;U)} &= \int_0^t (u(\tau), v(\tau))_U \, d\tau \\ &= \int_0^t \left(u(\tau), \sum_{k=1}^{n} v_k \, e^{i\omega_k \tau} \right)_U d\tau \\ &= \sum_{k=1}^{n} \left(\int_0^t u(\tau) \, e^{-i\omega_k \tau} \, d\tau, v_k \right)_U \\ &= \sum_{k=1}^{n} (\hat{u}_k^t, v_k)_U \\ &= ((\hat{u}_k^t), (v_k))_{l^2(U)}\end{aligned}$$

So we have

$$\begin{aligned}|((\hat{u}_k^t), (v_k))_{l^2(U)}| &= |(u,v)_{L^2(0,t;U)}| \\ &\leq \|u\|_{L^2(0,t;U)} \, \|v\|_{L^2(0,t;U)} \\ &\leq \|u\|_{\mathcal{U}} \, \|v\|_{\mathcal{U}} \\ &\leq \frac{1}{\sqrt{\zeta}} \, \|u\|_{\mathcal{U}} \, \|(v_k)\|_{l^2(U)}\end{aligned}$$

where ζ does not depend on n, and, by a density argument,

$$\|(\hat{u}_k^t)\|_{l^2(U)} \leq c \, \|u\|_{\mathcal{U}} \,, \quad \forall t \in [0,T]$$

with $c = 1/\sqrt{\zeta}$ ■

Let us then prove the following inequality:

$$\|x_n(t)\|_X \leq c_x \left(\|x_0\|_X + \|u\|_{\mathcal{U}} \right) \,, \quad \forall \, t \in [0,T] \,, \tag{3.4}$$

for all $n > 0$, where the constant c_x does not depend on n.

Because of (2.4),

$$\begin{aligned}\|x_n^a(t)\|_X^2 &= \sum_{k=1}^{n} [\omega_k^2 \, (y_0, \phi_k)_H^2 + (y_1, \phi_k)_H^2] \\ &= \|x_n(0)\|_X^2 \\ &\leq \|x_0\|_X^2 \,, \quad \forall \, t \in [0,T] \,.\end{aligned}$$

Besides,

$$\begin{aligned} \|x_n^c(t)\|_X^2 &= \sum_{k=1}^{n} \left| \int_0^t (u(\tau), b_k)_U \, e^{-i\omega_k \tau} \, d\tau \right|^2 \\ &\leq \beta^2 \sum_{k=1}^{n} |\hat{u}_k^t|^2 \,, \text{ by hypothesis (3.2)}, \\ &\leq \beta^2 \, \|(\hat{u}_k^t)_{k>0}\|^2_{l^2(U)} \,, \\ &\leq \beta^2 c^2 \, \|u\|^2_{\mathcal{U}} \,, \text{ because of Lemma 3.1}. \end{aligned}$$

We therefore obtain inequality (3.4), with $c_x = \max(1, \beta\, c) = \max(1, \beta/\sqrt{\zeta})$, which does not depends on n.

One can find in [16, 9.3 and 9.4] some theorems on the existence and uniqueness of a solution x of (2.2), $x \in L^\infty(0,T; H \times V')$, for $x_0 \in H \times V'$ and $u \in \mathcal{U}$, based upon the fact that x is the limit in $L^\infty([0,T]; H \times V')$ weak-star of a subsequence (x_μ) of (x_n).
In the same way, because of inequality (3.4), if $x_0 \in X$ and $u \in \mathcal{U}$, one can extract a subsequence (x_μ) of (x_n) such that:
(x_μ) converges weakly-star in $\mathcal{X}$ toward a limit $x \in \mathcal{X}$.
The fact that such an x is the unique solution of (2.2) is due to a theorem in [16, 9.4]. This implies that Λ_x is well define as a mapping from $X \times \mathcal{U}$ into $\mathcal{X}$.
By construction

$$\|x\|_{\mathcal{X}} \leq \lim_{\mu \to \infty} \|x_\mu\|_{\mathcal{X}} \,,$$

and inequality (3.4) implies that

$$\|x\|_{\mathcal{X}} \leq c_x \left(\|x_0\|_X + \|u\|_{\mathcal{U}} \right). \tag{3.5}$$

Inequality (3.5) leads in fact to a stronger regularity in time of the solution. Indeed, it implies $x \in C^0(0,T;X)$, and the strong convergence $x_\mu(t) \to x(t)$ in X, for any $t \in [0,T]$. More details are given in [14].
This proves part *(i)* of Theorem 3.1.

Part (ii)

Let us prove now the following inequality concerning the approximation of the output:

(3.6) $\qquad \|z_n\|_{\mathcal{Y}} \leq c_z\,(\|x_0\|_X + \|u\|_{\mathcal{U}})\,.$

for all $n > 0$, where the constant c_z does not depend on n.

We have

$$\begin{aligned} z_n^a(t) &= \mathcal{C}\,x_n^a(t) \\ &= \sum_{k=1}^{n}[-\omega_k\,(y_0,\phi_k)_H\,\sin\omega_k t + (y_1,\phi_k)_H\,\cos\omega_k t]\,c_k \\ &= \mathrm{Re}\sum_{k=1}^{n} e^{-i\omega_k t}\,[(y_1,\phi_k)_H - i\,\omega_k\,(y_0,\phi_k)_H]\,c_k \end{aligned}$$

and, using once again Theorem 2.1 and hypothesis (3.2),

$$\begin{aligned} \|z_n^a\|_{\mathcal{Y}} &\leq \frac{\gamma}{\sqrt{\zeta}}\,(\sum_{k>0}|(y_1,\phi_k)_H - i\,\omega_k\,(y_0,\phi_k)_H|^2)^{1/2} \\ &= \frac{\gamma}{\sqrt{\zeta}}\,\|x_0\|_X\,. \end{aligned}$$

Furthermore, by hypothesis (3.3), there exists a $C > 0$ such that

$$\|z_n^c\|_{\mathcal{Y}} \leq C\|u\|_{\mathcal{U}}$$

This proves inequality (3.6), with $c_z = \max(\gamma/\sqrt{\zeta}; C)$.

Now, because of (3.6), we may choose the subsequence (x_μ) used in the proof of part *(i)* of the theorem, such that:
$(z_\mu) = (\mathcal{C}\,x_\mu)$ converges weakly in $\mathcal{Y}$ toward a limit $\xi \in \mathcal{Y}$,
which satisfies by construction

$$\|\xi\|_{\mathcal{Y}} \leq \lim_{\mu\to\infty}\|z_\mu\|_{\mathcal{Y}}\,.$$

Then, inequality (3.6) implies

(3.7) $\qquad \|\xi\|_{\mathcal{Y}} \leq c_z\,(\|x_0\|_X + \|u\|_{\mathcal{U}})\,.$

In order to prove that $z = \xi$, we show now that the output $z = \mathcal{C}\,x$ is well defined for any initial data in a dense subspace of $X \times U$. However, the regularity property given for the solution x by part *(i)* of this theorem does not allow us to apply the output operator $\mathcal{C}$, which is not defined on such a solution. So, we have to establish a stronger regularity property on x to define the observation z. Denote

$$\mathcal{U}^1_* = H^1_*(0,T;U) = \{u \in H^1(0,T;U) \,/\, u(0) = 0\}$$

and

$$\begin{array}{rcl} \mathcal{S}_0 & = & \Lambda_x(\mathcal{D}(\mathcal{A}) \times \mathcal{U}^1_*) \\ \mathcal{S} & = & \Lambda_x(X \times \mathcal{U}) \end{array}$$

spaces of trajectories of (2.2), which satisfy $\mathcal{S}_0 \subset \mathcal{S}$, with dense embedding. Since $z = \mathcal{C}\,x = C\,\dot{y}$, we have to prove that the second component of x, i.e. $\dot{y}$, is sufficiently smooth, in order to give a meaning to the output ($(y, \dot{y})$ is the solution of the second order system (2.1)).

Lemma 3.2 $\mathcal{S}_0 \subset L^\infty(0,T;V \times V)$

Proof of Lemma 3.2

Let $x_0 = (y_0, y_1) \in \mathcal{D}(\mathcal{A}) \subset X$, and $u \in \mathcal{U}^1_* \subset \mathcal{U}$.
Let $x = (y, \dot{y}) = \Lambda_x(x_0, u)$ the associated solution of (2.2), or of the equivalent second order system (2.1). By part *(i)* of Theorem 3.1, we have

$$\mathcal{S}_0 \subset \mathcal{S} \subset \mathcal{X} = L^\infty(0,T;V \times H)$$

so that $y \in L^\infty(0,T;V)$.
Let us show that $\dot{y} \in L^\infty(0,T;V)$. For the autonomous part,

$$\dot{y}^a_n(t) = \sum_{k=1}^{n} [-(y_0, \phi_k)_H\, \omega_k \sin \omega_k t + (y_1, \phi_k)_H \cos \omega_k t]\, \phi_k$$

and

$$\begin{array}{rcl} \|\dot{y}^a_n(t)\|^2_V & \leq & 2 \sum_{k=1}^{n} \left[\omega_k^4\, |(y_0, \phi_k)_H|^2 + \omega_k^2\, |(y_1, \phi_k)_H|^2\right] \\ & = & 2\, \|\pi_n\, A\; y_0\|^2_H + 2\, \|\pi_n\, y_1\|^2_V \\ & \leq & 2\, \|A\; y_0\|^2_H + 2\, \|y_1\|^2_V \\ & = & 2\, \|x_0\|^2_{\mathcal{D}(\mathcal{A})} \end{array}$$

For the controlled part,

$$\dot{y}^c_n(t) = \sum_{k=1}^{n} \left[(\dot{u}(\tau), b_k)_U * \frac{\sin \omega_k \tau}{\omega_k}\right](t)\, \phi_k$$

since $u(0) = 0$; so

$$\begin{aligned}
\|\dot{y}_n^c(t)\|_V^2 &= \sum_{k=1}^{n} |[(\dot{u}(\tau), b_k)_U * \sin\omega_k\tau](t)|^2 \\
&= \sum_{k=1}^{n} |\int_0^t (\dot{u}(\tau), b_k)_U \sin\omega_k(t-\tau)\, d\tau|^2 \\
&\leq \sum_{k=1}^{n} |\int_0^t (\dot{u}(\tau), b_k)_U\, e^{-i\omega_k\tau}\, d\tau|^2 \\
&\leq \sum_{k=1}^{n} |\int_0^t \dot{u}(\tau)\, e^{-i\omega_k\tau}\, d\tau|^2\, \|b_k\|_U^2 \\
&\leq \beta^2\, \|(\hat{\dot{u}}_k^t)_{k>0}\|_{l^2(U)}^2, \text{ by (3.2).}
\end{aligned}$$

By assumption, $\dot{u} \in \mathcal{U}$ and Lemma 3.1 implies

$$\|\dot{y}_n^c(t)\|_V^2 \leq c^2\, \beta^2\, \|\dot{u}\|_{\mathcal{U}}^2.$$

Finally, for all $t \in [0, T]$

$$\|\dot{y}_n(t)\|_V \leq c_{\dot{y}} \left(\|x_0\|_{\mathcal{D}(\mathcal{A})} + \|\dot{u}\|_{\mathcal{U}}\right). \tag{3.8}$$

Since (x_n) converges weakly-star toward x in $\mathcal{X}$, $(\dot{y}_n)$ converges weakly-star toward $\dot{y}$ in $L^\infty(0, T; H)$ $(n = \mu)$. By inequality (3.8), $(\dot{y}_n(t))$ is bounded in V for $x_0 \in \mathcal{D}(\mathcal{A})$, $u \in \mathcal{U}_*^1$. So we may extract another subsequence (y_ν) such that $(\dot{y}_\nu)$ converges weakly-star toward $\dot{y}$ in $L^\infty(0, T; V)$, and

$$\|\dot{y}\|_{L^\infty(0,T;V)} \leq c_{\dot{y}} \left(\|x_0\|_{\mathcal{D}(\mathcal{A})} + \|\dot{u}\|_{\mathcal{U}}\right)$$

■

Since C is well-defined and continuous on V, it follows that $\mathcal{C} = (0\ C)$ is well defined on $\mathcal{S}_0$.

We show now that $\Gamma_\xi = \Lambda_z$ as a mapping from $\mathcal{D}(\mathcal{A}) \times \mathcal{U}_*^1$ into $\mathcal{Y}$. For any $v \in C^0(0, T; Y)$, and any $x_0 \in \mathcal{D}(\mathcal{A})$, $u \in \mathcal{U}_*^1$,

$$\begin{aligned}
(\xi, v)_{\mathcal{Y}} = \lim_{\nu\to\infty} (z_\nu, v)_{\mathcal{Y}} &= \lim_{\nu\to\infty} (C\, \dot{y}_\nu, v)_{\mathcal{Y}} \\
&= \lim_{\nu\to\infty} \int_0^T \langle \dot{y}_\nu(t), C^*\, v(t)\rangle_{VV'}\, dt
\end{aligned}$$

Inequality (3.8) and the weak-star convergence of $(\dot{y}_\nu)$ toward $\dot{y}$ in $L^\infty(0, T; V)$ enable us to apply the Lebesgue theorem:

$$\begin{aligned}
(\xi, v)_{\mathcal{Y}} &= \int_0^T \langle \dot{y}(t), C^* v(t) \rangle_{VV'}\, dt \\
&= (C\,\dot{y}, v)_{\mathcal{Y}} \\
&= (\mathcal{C}\,x, v)_{\mathcal{Y}} \\
&= (z, v)_{\mathcal{Y}}
\end{aligned}$$

Then, as $C^0(0,T;Y)$ is dense in $\mathcal{Y}$, $z = \xi = C\,\dot{y} = \mathcal{C}\,x \in \mathcal{Y}$ for all $x \in \mathcal{S}_0$, or $\Lambda_z = \Gamma_\xi$ on $\mathcal{D}(\mathcal{A}) \times \mathcal{U}^1_*$, and (3.7) gives

$$(3.9) \qquad \|z\|_{\mathcal{Y}} \leq c_z \left(\|x_0\|_X + \|u\|_{\mathcal{U}}\right) ,$$

for all $x_0 \in \mathcal{D}(\mathcal{A})$, $u \in \mathcal{U}^1_*$.
Let us prove now that the output z is well defined in $\mathcal{Y}$ for any initial data $x_0 \in X$ and any control $u \in \mathcal{U}$. In other words we show that an output $z \in \mathcal{Y}$ is associated to any solution $x \in \mathcal{S}$.
We have just seen that this is the case for $x \in \mathcal{S}_0$. From the first inequality (3.5), the semi-norm $\|x\|_{\mathcal{S}} = \|x_0\|_X + \|u\|_{\mathcal{U}}$ is a norm on $\mathcal{S}$. Moreover, $\mathcal{S}_0 \subset \mathcal{S}$ with dense and continuous embedding for $\|\ \|_{\mathcal{S}}$ and, for all $x \in \mathcal{S}_0$, (3.9) implies

$$\|z\|_{\mathcal{Y}} = \|\mathcal{C}\,x\|_{\mathcal{Y}} \leq c_z \|x\|_{\mathcal{S}}$$

so that $\mathcal{C}$ is continuous on $\mathcal{S}_0$ equiped with the norm $\|\ \|_{\mathcal{S}}$.
By density, we can extend $\mathcal{C}$ into an operator (still denoted by $\mathcal{C}$) continuous on $\mathcal{S}$. The observation z is then well defined in $\mathcal{Y}$, and satisfies (3.9) for any $x_0 \in X$, $u \in \mathcal{U}$: the mapping Λ_z is continuous. This ends the proof of Theorem 3.1 ■

Remark 3.1 The operator $\mathcal{C}$ is first defined on $\mathcal{S}_0$ and then extended into an operator on $\mathcal{S}$ by application of the Hahn-Banach theorem (although it may not be defined on X), and the following diagram commutes:

$$
\begin{tikzcd}
\begin{matrix} \mathcal{D}(\mathcal{A}) \times \mathcal{U}^1_* \\ \cap \\ X \times \mathcal{U} \end{matrix} \arrow[r, "\Lambda_y"] \arrow[dr, "\Lambda_z"'] & \mathcal{S}_0 \subset \mathcal{S} \subset C^0(0,T;X) \arrow[d, "\mathcal{C}"] \\
& \mathcal{Y}
\end{tikzcd}
$$

Remark 3.2 Ho and Russell[11] give sufficient conditions for the regularity of solutions of open-loop systems as the state evolution equation in (2.2), but do not consider the problem of existence of an unbounded output.
Recently, Curtain and Weiss[9] studied unbounded observations associated to particular systems (those of Salamon class), and showed that it is well-defined under some sufficient conditions concerning the Laplace transform of the input-ouput mapping. However, the Euler-Bernoulli beam we consider as an application may not be modelled by such a Salamon system, since it is equiped with colocated sensors and actuators (see Curtain[8]).

3.2 Observability

We give here a sufficient condition for strong observability of system (2.2). Consider the autonomous system

$$
(3.10) \qquad \begin{cases} \dot{x}(t) = \mathcal{A}\, x(t) \\ z = \mathcal{C}\, x \\ x(0) = x_0 \end{cases}
$$

Theorem 3.2 *Under hypotheses* (3.1), (3.2) *of Theorem* 3.1, *and if there exists a constant* $\gamma' > 0$ *such that*

$$
(3.11) \qquad \|c_k\|_Y \geq \gamma' \ , \quad \forall k > 0\,,
$$

then for all $x_0 \in X$, *the output* z *of the autonomous system* (3.10) *satisfies*

$$
(3.12) \qquad \|z\|_{\mathcal{Y}} \geq c_o\, \|x_0\|_X
$$

for a constant $c_o > 0$, *and for every*

$$T > T_0 = \frac{2\pi}{\alpha}.$$

System (3.10) *is observable* [1] *for any time* $T > T_0$.

Remark 3.3 Time T_0 from which system (3.10) is observable becomes arbitrarily short when

$$\lim_{k\to\infty} \omega_{k+1} - \omega_k = +\infty.$$

Proof of Theorem 3.2

The output of system (3.10) is given by

$$\begin{aligned} z(t) &= \sum_{k\geq 1} [-(y_0, \phi_k)_H\, \omega_k \sin \omega_k t + (y_1, \phi_k)_H \cos \omega_k t]\; c_k \\ &= \frac{1}{2} \sum_{\substack{k\in Z \\ k\neq 0}} e^{i\omega_k t}\, [(y_1, \phi_k)_H - i\,\omega_k\, (y_0, \phi_k)_H]\; c_k \end{aligned}$$

If $T > T_0 = 2\pi/\alpha$, applying inequality (2.7) of Theorem 2.1, we obtain

$$\begin{aligned} \|z\|^2_{\mathcal{Y}} &\geq \frac{1}{4\eta} \sum_{\substack{k\in Z \\ k\neq 0}} |(y_1, \phi_k)_H - i\,\omega_k\,(y_0, \phi_k)_H|^2\, \|c_k\|^2_Y \\ &\geq \frac{\gamma'^2}{4\eta}\, \|x_0\|^2_X \end{aligned}$$

by hypothesis (3.11). This proves Theorem 3.2, with $c_o = \gamma' / 2\sqrt{\eta}$ ■

4 Closed-loop systems

In this section we suppose that hypotheses of Theorems (3.1) and (3.2) are satisfied.

We study the class of closed-loop systems obtained by introducing in (2.2) a linear output feedback. We prove that these systems are well posed, and remain observable. Due to the fact that the feedback is constructed from the observation (and not from the whole state), this observability property leads to a proof of uniform exponential stability of such linear closed-loop systems.

[1] The observability is taken here in the sense of Russell, [19, p.645]

4.1 Well-posedness

Suppose now that the finite dimensional input and output spaces U and Y are equal, so that $\mathcal{U} = \mathcal{Y}$. We consider here the case where $B = C^*$ (and $\mathcal{B} = \mathcal{C}^*$). System (2.2) becomes

$$(4.1) \qquad \begin{cases} \dot{x}(t) = \mathcal{A}\, x(t) + \mathcal{C}^*\, u(t) \\ z = \mathcal{C}\, x \\ x(0) = x_0 \end{cases}$$

Let $x_0 \in X$, $u \in \mathcal{Y}$. Let $K \in \mathcal{L}(Y)$ be a linear mapping. Denote $\kappa = \|K\|_{\mathcal{L}(Y)}$. We study properties of the output feedback

$$(4.2) \qquad u_K = -K\, z(u).$$

where $z(u)$ is the ouptut of (4.1) associated to the input u.
Let $v \in \mathcal{Y}$ and define the mapping

$$\psi_{v,K} : u \mapsto -K\, z(u) + v\,,$$

Lemma 4.1 *If κ is small enough, then $\psi_{v,K}$ admits a unique fixed point $v_K \in \mathcal{Y}$.*

Proof of Lemma 4.1

First, $\psi_{v,K}$ maps $\mathcal{Y}$ into $\mathcal{Y}$:

$$\begin{aligned} \|\psi_{v,K}(u)\|_{\mathcal{Y}} &= \|-K\, z(u) + v\|_{\mathcal{Y}} \\ &\leq \kappa\, \|z(u)\|_{\mathcal{Y}} + \|v\|_{\mathcal{Y}} \\ &\leq \kappa\, c_z\, (\|u\|_{\mathcal{Y}} + \|x_0\|_X) + \|v\|_{\mathcal{Y}} \end{aligned}$$

from inequality (3.9), and Theorem 3.1. So, for all $u \in \mathcal{Y}$, $\psi_{v,K}(u) \in \mathcal{Y}$.
Moreover, if κ is small enough, $\psi_{v,K}$ is a contraction on $\mathcal{Y}$:

$$\|\psi_{v,K}(u_1) - \psi_{v,K}(u_2)\|_{\mathcal{Y}} \leq \kappa\, \|z_1 - z_2\|_{\mathcal{Y}} \leq \kappa\, c_z\, \|u_1 - u_2\|_{\mathcal{Y}}$$

for inputs u_1, $u_2 \in \mathcal{Y}$, z_1 and z_2 denoting the corresponding outputs of (4.1), still applying (3.9). Hence, if $\kappa < 1/c_z$, $\psi_{v,K}$ is a contraction and admits a unique fixed point $v_K = -K\, z(v_K) + v$ in $\mathcal{Y}$ ■

Suppose that indeed $\kappa < 1/c_z$.
By Lemma 4.1, and Theorem 3.1, the system:

$$(4.3)\qquad \begin{cases} \dot{x}(t) = (\mathcal{A} - \mathcal{C}^* K \mathcal{C})\, x(t) + \mathcal{C}^* v(t) \\ z = \mathcal{C}\, x \\ x(0) = x_0 \end{cases}$$

obtained after introducing the feedback $u = v_K$ in (4.1), is well posed and admits a unique solution $x \in \mathcal{X}$, and an output $z \in \mathcal{Y}$.
Consider now the closed-loop system obtained by taking $v(t) \equiv 0$ in (4.3),

$$(4.4)\qquad \begin{cases} \dot{x}(t) = (\mathcal{A} - \mathcal{C}^* K \mathcal{C})\, x(t) \\ z = \mathcal{C}\, x \\ x(0) = x_0 \end{cases}$$

Remark 4.1 The hypothesis κ small, used to establish the well-posedness of the open-loop (4.3), is no longer necessary for the well-posedness of the autonomous system (4.4). Indeed, it is sufficient to assume that K is non-negative : in this case the operator $\mathcal{A} - \mathcal{C}^* K \mathcal{C}$ is maximal monotoneous and Hille-Yosida Theorem applies (see [2] for instance).

4.2 Observability

We show that, as a consequence of the observability of system (3.10), the closed-loop system (4.4) is observable.

Lemma 4.2 *If κ is small enough, the closed-loop system* (4.4) *is observable from time $T_0 = 2\,\pi/\alpha$.*

Proof of Lemma 4.2

Denote by z^0 the output of the autonomous system (3.10) corresponding to initial data x_0, and by z the output of the closed-loop system (4.4), for the same initial condition. We have

$$\|z - z^0\|_{\mathcal{Y}} \geq \|z^0\|_{\mathcal{Y}} - \|z\|_{\mathcal{Y}}$$

and, because of (3.9), as $z - z^0$ is the output of a system like (2.2) with zero initial conditions and a control $u = K\, z$,

$$\begin{aligned} \|z - z^0\|_{\mathcal{Y}} &\leq c_z\, \|K\, z\|_{\mathcal{Y}} \\ &= c_z\, \kappa\, \|z\|_{\mathcal{Y}} \end{aligned}$$

So

$$c_z\, \kappa\, \|z\|_{\mathcal{Y}} \geq \|z^0\|_{\mathcal{Y}} - \|z\|_{\mathcal{Y}}$$

and

$$\|z\|_{\mathcal{Y}} \geq \frac{1}{1+\kappa\, c_z} \|z^0\|_{\mathcal{Y}}$$

By Theorem 3.2, for $T > T_0$, (3.10) is observable and there exists a constant $c_o > 0$ such that

$$\|z^0\|_{\mathcal{Y}} \geq c_o \, \|x_0\|_X$$

Hence

$$\|z\|_{\mathcal{Y}} \geq \frac{c_o}{1+\kappa\, c_z} \, \|x_0\|_X = C_o \, \|x_0\|_X$$

where $C_o = c_o/(1+\kappa\, c_z) > 0$ ■

4.3 Exponential stability

Let

$$E(t) = \frac{1}{2} \, \|x(t)\|_X^2$$

the energy of a solution $x(t)$ of (4.4) at time t. We show that, under a positivity assumption on the gain K, observability of system (4.4) implies exponential stability. This follows from the fact that the feedback is constructed from the observation. The link between observability of a conservative system and exponential decay for the solutions of the same system with dissipation has already been pointed out. See for example Haraux[10] and Zuazua[20].

Theorem 4.1 *If the feedback operator $K \in \mathcal{L}(\mathcal{Y})$ is such that κ is small enough and*

$$(K\,\xi, \xi)_Y \geq k \, \|\xi\|_Y^2 \ , \ k > 0 \ , \ \forall \xi \in Y \, ,$$

then the energy of the closed-loop system (4.4) decays exponentially: there exists a constant $\mu > 0$ such that for all $t > 0$,

$$E(t) \leq E(0)\, e^{-\mu t} \, .$$

Proof of Theorem 4.1

From the expression of $E(t)$, we compute formally the energy variation between times 0 and T:

$$E(T) - E(0) = \int_0^T \langle x(t), \dot{x}(t) \rangle_{XX'}\, dt$$

which gives in the case of system (4.4) and operator $\mathcal{A}$,

$$E(T) - E(0) = -(K\, z, z)_{\mathcal{Y}}$$

so that

$$E(0) - E(T) \geq k\, \|z\|_{\mathcal{Y}}^2$$

by hypothesis. However, by Lemma 4.2, system (4.4) is observable for $T > T_0$, and

$$\|z\|_{\mathcal{Y}} \geq C_o \|x_0\|_X$$

for $C_o > 0$. Hence,

$$E(0) - E(T) \geq k\, C_o^2\, \|x_0\|_X^2 = 2\, k\, C_o^2\, E(0)$$

and

$$E(T) \leq (1 - 2\, k\, C_o^2)\, E(0)\,.$$

But, $\forall t > 0, \exists\, p \in N$ such that $p\,T \leq t < (p+1)\,T$, and

$$E(t) \leq E(p\,T) \leq (1 - 2\, k\, C_o^2)^p\, E(0).$$

Let

$$\mu = -\frac{1}{2\,T} \ln(1 - 2\, k\, C_o^2) > 0$$

We have

$$E(t) \leq E(0)\, e^{-(p+1)\mu T} \leq E(0)\, e^{-\mu t}\,, \forall t > 0$$

which proves the uniform exponential decay of the energy ∎

Remark 4.2 The exponential decay rate μ of the energy is explicitly given in term of constants involved in the behaviour of system (2.2) and of feedback operator K by

$$\mu = -\frac{1}{2\,T} \ln(1 - \frac{2\, k\, c_o^2}{(1 + \kappa\, c_z)^2})$$

Recall that c_o denotes the observability constant of open-loop system (2.2), while c_z is the continuity constant of mapping Λ_z.

Remark 4.3 The output feedback $u_K = -K\,z$ is optimal for the criterion

$$J(u) = \frac{1}{2}\left((K\,z,z)_{\mathcal{Y}} + (K^{-1}\,u,u)_{\mathcal{Y}}\right) + E(T)$$

By Theorem 3.1, if $u \in \mathcal{Y}$, $J(u)$ makes sense for the system (2.2), and

$$J(u_K) = E(0) \leq J(u)$$

for all $u \in \mathcal{Y}$.
As it was pointed out by Lasiecka and Triggiani[13] and Pritchard and Salamon[17], the optimal control problem is closely linked with the regularity problem for the open-loop system. Moreover, the results of Zuazua in [21] may be used to show that this feedback remains optimal in case of infinite horizon.

Remark 4.4 Stability properties depending on an observability assumption, with estimations of the decay rate, have also been established for a class of sublinear output feedbacks, in collaboration with Conrad, in [6], [7]. These results depend also on a continuity hypothesis of mapping (Λ_x, Λ_z) related to the open-loop system. Theorem 3.1 provides sufficient conditions for this well-posedness assumption to be satisfied.

5 Application to an Euler-Bernoulli beam

Consider a beam which is clamped at one end and controlled by a point force at the other end. One observes the transverse velocity at the controlled end.

Let y be the lateral displacement of the beam. This system can be modelled by the following Euler-Bernoulli equation

$$(5.1)\qquad \begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) + \dfrac{\partial^4 y}{\partial x^4}(x,t) = 0 & x \in \,]0,1[\,,\ t > 0 \\[2ex] y(0,t) = \dfrac{\partial y}{\partial x}(0,t) = 0 & \text{clamped end} \\[2ex] \dfrac{\partial^2 y}{\partial x^2}(1,t) = 0\ ,\ \dfrac{\partial^3 y}{\partial x^3}(1,t) = u(t) & \text{controlled end} \\[2ex] z(t) = \dfrac{\partial y}{\partial t}(1,t) & \text{transverse velocity} \end{cases}$$

with initial data $y(x,0) = y_0(x)$ and $\frac{\partial y}{\partial t}(x,0) = y_1(x)$. Let us define

$$\begin{array}{rcl} \Omega & = &]0,1[\\ H & = & L^2(\Omega) \\ V & = & \{v \in H^2(\Omega)/v(0) = \frac{\partial v}{\partial x}(0) = 0\} \\ U & = & Y \;=\; \mathrm{R} \;\; (\mathcal{U} = \mathcal{Y} = L^2(0,T)) \end{array}$$

and for all $v, w \in V$

$$\langle A\,v, w\rangle_{V'V} = \int_\Omega \frac{\partial^2 v}{\partial x^2}(s)\,\frac{\partial^2 w}{\partial x^2}(s)\,ds$$

$$Cv = v(1)$$

With these notations, (5.1)

$$(5.2) \qquad \begin{cases} \ddot{y}(t) + A\,y(t) = C^*\,u(t) \\ z(t) = C\,\dot{y}(t) \\ y(0) = y_0,\;\; \dot{y}(0) = y_1 \end{cases}$$

or, with the notations of Section 1,

$$(5.3) \qquad \begin{cases} \dot{x}(t) = \mathcal{A}\,x(t) + \mathcal{C}^*\,u(t) \\ z(t) = \mathcal{C}\,x(t) \\ x(0) = x_0 \end{cases}$$

We verify that operators A and C satisfy the hypotheses of Theorems 3.1 and 3.2.

Verification of (3.1)

We have[2]

$$\omega_k = \alpha_k^2$$

where the α_k are the positive solutions of $\cos\alpha\,\mathrm{ch}\,\alpha + 1 = 0$ and take the form

$$\alpha_k = (k - \frac{1}{2})\,\pi + \sigma_k$$

with $|\sigma_k| < \pi/2$ and $|\sigma_k| \searrow 0$ as $k \to \infty$.
Then, (3.1) is true, with an asymptotic gap $\alpha = \infty$.

Verification of (3.2) and (3.11)

The normalized eigenfunctions of operator A are given by

[2]Expressions and properties of α_k and ϕ_k, $k > 0$, are given by Ball and Slemrod[1]

$$\phi_k(x) = \operatorname{ch} \alpha_k x - \cos \alpha_k x - \gamma(\alpha_k)(\operatorname{sh} \alpha_k x - \sin \alpha_k x)$$

where

$$\gamma(\alpha) = \frac{\operatorname{ch} \alpha + \cos \alpha}{\operatorname{sh} \alpha + \sin \alpha}$$

so that $b_k = c_k = \phi_k(1) = 2(-1)^{k-1}$ and $|c_k| = 2$: (3.2) and (3.11) are satisfied with $\beta = \gamma = \gamma' = 2$.

Verification of (3.3)

We show that there exists a constant $c > 0$ such that

$$\|z_n^c\|_{L^2(0,T)} \leq c\,\|u\|_{L^2(0,T)}\ ,\ \forall n > 0 \tag{5.1}$$

We have

$$\begin{aligned}
z_n^c(t) &= \sum_{k=1}^{n} b_k^t\, c_k [u(\tau) * \cos \omega_k \tau](t) \\
&= \frac{1}{2} \sum_{\substack{k=-n \\ k \neq 0}}^{n} b_k^t\, c_k\, [u(\tau) * e^{i\omega_k \tau}](t) \\
&= \frac{1}{2} [u * h_n^T](t)
\end{aligned}$$

where

$$h_n^T(\tau) = \begin{cases} \sum\limits_{\substack{k=-n \\ k \neq 0}}^{n} b_k^t\, c_k\, e^{i\omega_k \tau} = 4 \sum\limits_{\substack{k=-n \\ k \neq 0}}^{n} e^{i\omega_k \tau} & 0 \leq \tau \leq T \\ 0 & \tau > T\,,\ \tau < 0 \end{cases}$$

Denote by $\hat{h}_n^T$ the Fourier transform of h_n^T. Then,

$$\begin{aligned}
2\,\|z_n^c\|_{L^2(0,T)} &= \|u * h_n^T\|_{L^2(0,T)} \\
&\leq \|u * h_n^T\|_{L^2(\mathbb{R})} \\
&= \|\hat{u}\, \hat{h}_n^T\|_{L^2(\mathbb{R})} \\
&\leq \|\hat{u}\|_{L^2(\mathbb{R})}\, \|\hat{h}_n^T\|_{L^\infty(\mathbb{R})} \\
&\leq \|u\|_{L^2(0,T)}\, \|\hat{h}_n^T\|_{L^\infty(\mathbb{R})} \\
&\leq c\, \|u\|_{L^2(0,T)}
\end{aligned}$$

if c is such that

$$(5.5) \qquad \|\hat{h}_n^T\|_{L^\infty(I\!R)} \le \frac{c}{2}$$

for all $n > 0$. System (5.3) satisfies (5.5), due to the growth to infinity of the ω_k, which is sufficiently fast to make the truncated transfer functions $\hat{h}_n^T$ uniformly bounded in $L^\infty(I\!R)$. A detailed proof of this property may be found in [14] and [15]. This makes (3.3) true for system (5.3).

Then, Theorem 3.1 applies and shows that for all $x_0 \in X$ and for all $u \in \mathcal{U}$, there exists a unique solution x of (5.3) such that

$$x \in \mathcal{X} \text{ and } z \in \mathcal{Y},$$

and the mapping $(x_0 \, , \, u) \to (x \, , \, z)$ is continuous. This implies the well-posedness of open-loop system (5.1) and proves the existence in $L^2(0,T)$ of the point boundary observation z, for any control u in $L^2(0,T)$.
Moreover, since operator C satisfies also hypothesis (3.11), Theorem 3.2 gives the observability of (5.3) from an arbitrary short time.

Let $K > 0$. The closed loop-system obtained by taking $u(t) = -K\, z(t) + v(t)$ has the same form as (4.3) and corresponds to the following explicit system

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) + \dfrac{\partial^4 y}{\partial x^4}(x,t) = 0 \quad x \in]0,1[\, , \, t > 0 \\[2ex] y(0,t) = \dfrac{\partial y}{\partial x}(0,t) = 0 \\[2ex] \dfrac{\partial^2 y}{\partial x^2}(1,t) = 0 \; , \; \dfrac{\partial^3 y}{\partial x^3}(1,t) = -K\, \dfrac{\partial y}{\partial t}(1,t) + v(t) \\[2ex] z(t) = \dfrac{\partial y}{\partial t}(1,t) \end{cases}$$

For K small enough, and every $v \in L^2(0,T)$, this system is well posed. Moreover, for $v = 0$, Theorem 4.1 implies exponential decrease of energy.
This corresponds to a particular case of a result obtained by by Chen, Delfour, Krall, and Payre[4] for serially connected beams, which have also been studied by Conrad[5], and Rideau[18], to establish an estimation of the decay rate.

Regularity and stability results have recently been established for some more general systems composed of non-homogeneous connected beams, using multipliers methods, by Conrad, Leblond, and Marmorat (in preparation). These stability properties concern a class of non-linear output feedbacks and are established with estimations of the decay rate.

6 Extension to a wave equation

Consider here the following wave equation

$$
(6.6)\qquad \begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) - \dfrac{\partial^2 y}{\partial x^2}(x,t) = 0 \quad x \in]0,1[\,,\ t > 0 \\[2ex] y(0,t) = 0 \\[2ex] \dfrac{\partial y}{\partial x}(1,t) = u(t) \\[2ex] z(t) = \dfrac{\partial y}{\partial t}(1,t) \end{cases}
$$

which may model the torsion of a beam, y being the angle of the beam with its rest position. We observe the angular velocity at the end of the beam, and control by a point torque. Define here

$$
\begin{array}{ll} H = L^2(\Omega) & \Omega =]0,1[\\ V = \{y \in H^1(\Omega) \,/\, y(0) = 0\} & \\ U = Y = \mathrm{R} & (\mathcal{U} = \mathcal{Y} = L^2(0,T)) \end{array}
$$

and, for all $v, w \in V$,

$$
\langle A\, v, w\rangle_{V'V} = \int_\Omega \frac{\partial v}{\partial x}(s)\, \frac{\partial w}{\partial x}(s)\, ds
$$

$$
C\, v = v(1)\,.
$$

Let $y_0 \in V$, $y_1 \in H$. As in section 5, (6.6) can be rewritten as

$$
(6.7)\qquad \begin{cases} \ddot{y}(t) + A\, y(t) = C^*\, u(t) \\ z(t) = C\, \dot{y}(t) \\ y(0) = y_0,\ \dot{y}(0) = y_1 \end{cases}
$$

or, with the notations of Section 1,

$$
(6.8)\qquad \begin{cases} \dot{x}(t) = \mathcal{A}\, x(t) + \mathcal{C}^*\, u(t) \\ z(t) = \mathcal{C}\, x(t) \\ x(0) = x_0 \end{cases}
$$

We verify now that operators A and C satisfy hypotheses (3.1), (3.2), and (3.11) of Theorems 3.1 and 3.2. But hypothesis (3.3) is not verified here. However, we show by an explicit computation that the conclusion of Theorem

3.1 remains true.

State regularity: application of part *(i)* of Theorem 3.1

Verification of (3.1)

Eigenvalues of operator A are given by $\lambda_k = \omega_k^2$ for $k \geq 1$, where

$$\omega_k = (k - \frac{1}{2})\pi .$$

Then

$$\varliminf_{k\to\infty} \omega_{k+1} - \omega_k = \pi .$$

and (3.1) is true, with gap $\alpha = \pi$.

Verification of (3.2) and (3.11)

Eigenfunctions of A are given by

$$\phi_k(x) = \sqrt{2}\,\sin \omega_k\, x .$$

So

$$b_k = c_k = \phi_k(1) = (-1)^{k-1}\sqrt{2} ,$$

and $|c_k| = |b_k| = \sqrt{2}$. Then, 3.2 is true with $\beta = \gamma = \sqrt{2}$.

Then, part *(i)* of Theorem 3.1 holds, and system (6.8) admits a unique solution in $\mathcal{X}$. Moreover, the mapping Λ_x is continuous.

Existence of the output

For the wave equation (6.6), hypothesis (3.3) cannot be checked by the computation method used for the Euler-Bernoulli beam. This is due to the fact that eigenvalues of the wave operator statisfy $\omega_k \sim k$ and do not grow to infinity as fast as eigenvalues of the beam operator (which satisfy $\omega_k \sim k^2$).
However, here also, formulation (6.8) of the wave equation (6.6) leads to the following expression of the input-output mapping:

$$z^c(t) = \frac{1}{2}\,[u * h_\infty^T](t) \tag{6.9}$$

limit as $n \to \infty$ of the similar partial formulation page (5.4) related to the beam, where

$$h_\infty^T(\tau) = \begin{cases} \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} b_k^t c_k e^{i\omega_k \tau} = 2 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} e^{i\omega_k \tau} & 0 \leq \tau \leq T \\ 0 & \tau > T \text{ or } \tau < 0 \end{cases}$$

We derive here an explicit expression for the distribution h_∞^T. It provides an upper-bound for this truncated impulse response, as a convolution operator from $L^2(0,T)$ into $L^2(0,T)$, and, thereby, boundedness in $L^2(0,T)$ of the output z, and continuity of the input-output mapping are established.
Thus, although we have no information about the uniform boundedness of h_n^T, $\forall n > 0$, or wether hypothesis (3.3) holds, a direct computation shows that the conclusion of part *(ii)* of Theorem 3.1 remains true.

Computation of h_∞^T

In order to write h_∞^T explicitly for almost every $T > 0$, we first compute h_∞^∞, and then take its restriction to interval $[0,T]$. So

$$h_\infty^\infty(\tau) = \begin{cases} 2 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} e^{i\omega_k \tau} = 2 \sum_{k=-\infty}^{\infty} e^{i(k-\frac{1}{2})\pi\tau} & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

Define $\check{h}_\infty^\infty(\tau) = h_\infty^\infty(-\tau)$, for any $\tau \in I\!R$. Then, for any $\tau \in I\!R$, the following even distribution

$$h_\infty^\infty(\tau) + \check{h}_\infty^\infty(\tau) = 2 \sum_{k\in Z} e^{i(k-\frac{1}{2})\pi\tau} = 2\, e^{-i\frac{\pi}{2}\tau} \sum_{k\in Z} e^{ik\pi\tau}$$

can be computed by the Poisson formula

$$\frac{1}{t_0} \sum_{k\in Z} e^{2ik\pi\frac{\tau}{t_0}} = \sum_{k\in Z} \delta_{kt_0},$$

for any $t_0 > 0$. We obtain

$$h_\infty^\infty(\tau) + \check{h}_\infty^\infty(\tau) = 4\, e^{-i\frac{\pi}{2}\tau} \sum_{k\in Z} \delta_{2k} = 4 \sum_{k\in Z} (-1)^k \delta_{2k},$$

the 2-periodic alternating Dirac distribution.
Hence, h_∞^∞ is given by

$$h_\infty^\infty = \begin{cases} 2\,\delta + 4\,\sum\limits_{k\geq 1}(-1)^k\,\delta_{2k} & \text{on } [0,\infty[\\ \\ 0 & \text{on }]-\infty,0[\end{cases}$$

Then, let

$$N = N(T) = E(\frac{T}{2}),$$

where E denotes the integer part. Restriction h_∞^T of h_∞^∞ to the interval $[0,T]$ is given, for any $T \neq 2\,k$, $k \in \mathrm{Z}$, by the following distribution of $\mathcal{D}'(I\!R_+)$

$$(6.10) \qquad h_\infty^T = 2\,\delta + 4\,\sum_{k=1}^{N}(-1)^k\,\delta_{2k} \text{ for } t \geq 0$$

The point here is that there remain only a finite number of Dirac masses in the compact support of h_∞^T.

Computation of $\|\hat{h}_\infty^T\|_{L^\infty(\boldsymbol{R})}$

For any $\nu \in I\!R$, the Fourier transform $\hat{h}_\infty^T$ is given by the following trigonometric polynomial

$$\begin{aligned} \hat{h}_\infty^T(\nu) = &\ 2 + 4\,(-e^{-2i\pi\nu} + e^{-4i\pi\nu} + \cdots + (-1)^N e^{-2Ni\pi\nu}) \\ = &\ 2 + 4\,\sum_{k=1}^{N}(-1)^k e^{-2ik\pi\nu} \\ = &\ -2 + 4\,\sum_{k=0}^{N}(-1)^k e^{-2ik\pi\nu} \\ = &\ -2 + 4\,\frac{1-(-1)^{N+1}\,e^{-2i(N+1)\pi\nu}}{1+e^{-2i\pi\nu}}\,. \end{aligned}$$

A straightforward computation leads to

$$\sup_{\nu\in\boldsymbol{R}} |\hat{h}_\infty^T(\nu)| = |\hat{h}_\infty^T(\frac{1}{2} + k)| == -2 + 4\,(N+1)$$

for $k \in \mathrm{Z}$. Finally,

$$(6.11) \qquad \sup_{\nu\in\boldsymbol{R}} |\hat{h}_\infty^T(\nu)| = 4\,N + 2 = 4\,E(\frac{T}{2}) + 2$$

so that $\hat{h}_\infty^T$ is bounded in $L^\infty(I\!R)$.
We may not conclude to the uniform boundedness of $\hat{h}_n^T$, for any $n > 0$.

Remark 6.1 Boundedness in $L^\infty(I\!R)$ of the Fourier transform $\hat{h}_\infty^T$ follows then from compacity of the support of h_∞^T. This is false when $T \to \infty$, as the support of h_∞^∞ does not remain compact, and the transfer function $\hat{h}_\infty^\infty$ does not belong to $L^\infty(I\!R)$anymore.

By formulation (6.9) of the input-output mapping, expression (6.10) of h_∞^T gives

$$z^c(t) = 2\,u(t) + 4\sum_{k=1}^{N}(-1)^k\,u(t-2\,k)$$

so,

$$\|z^c(t)\|_{L^2(0,T)} \le (2+4\,N)\,\|u\|_{L^2(0,T)}\,. \tag{6.12}$$

Though we don't verify (3.3), inequality (6.12) - obtained by an explicit computation - implies continuity of the mapping

$$\mathcal{H}^T = \Lambda_z\big|_{L^2(0,T)} : \begin{array}{ccc} L^2(0,T) & \longrightarrow & L^2(0,T) \\ u & \longmapsto & z^c \end{array}$$

Remark 6.2 The bound $2+4\,N$, which occurs in expression (6.12), is reached by inputs u of the form

$$u(t) = \chi_{[0,\rho]}\;, \quad 0 \le \rho \le 2$$

so that $\|\mathcal{H}^T\| = 2+4\,N$ as a convolution operator in $\mathcal{L}(L^2(0,T))$. Hence, by (6.11), $\|\hat{h}_\infty^T\|_{L^\infty(I\!R)} = \|\mathcal{H}^T\|$, and this norm corresponds to the total mass of h_∞^T in $(L^\infty(0,T))'$.

The output z^a of the autonomous system (with zero input) associated to (6.8) depends only on x_0, and, as hypothesis (3.2) is verified

$$\|z^a\|_{L^2(0,T)} \le c_z^a\,\|x_0\|_X\,,$$

by Theorem 3.1. Hence, Λ_z is continuous.

Finally, the mapping

$$\Lambda = (\Lambda_y, \Lambda_z) : \begin{array}{ccc} X \times L^2(0,T) & \longrightarrow & \mathcal{X} \times L^2(0,T) \\ (x_0, u) & \longmapsto & (x, z) \end{array}$$

is continuous.
Recall that, concerning the state x, the continuity comes from the verification of (3.1), (3.2), and application of part *(i)* of Theorem 3.1, but concerning the output z, it proceeds from the obtention of an explicit expression (6.12) for the impulse response.

Since hypothesis (3.11) is true, Theorem 3.2 implies observability of system (6.8) from time $T_0 = 2$.

Consider then an output feedback $u = -K\,z + v$, for K small, $v \in L^2(0,T)$. This leads to a closed-loop system which can be written as (4.3) or equivalently as :

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(x,t) - \dfrac{\partial^2 y}{\partial x^2}(x,t) = 0 & x \in]0,1[\,,\ t > 0 \\ y(0,t) = 0 & \\ \dfrac{\partial y}{\partial x}(1,t) = -K\,\dfrac{\partial y}{\partial t}(1,t) + v(t) & \\ z(t) = \dfrac{\partial y}{\partial t}(1,t) & \end{cases}$$

By the continuity of Λ and the observability property, Theorem 4.1 applies to give uniform exponential stability of this feedback controlled wave equation when $v = 0$. This is a particular case of a result obtained by Chen[3].

Aknowledgments
We like to thank here F. Conrad, M. Pierre, and E. Zuazua for their very careful reading of this paper and their helpfull suggestions.

References

[1] J.M. Ball and M. Slemrod. Nonharmonic Fourier series and the stabilization of distributed semi-linear control systems. *Communications on Pure and Applied Mathematics*, 32:555–587, 1979.

[2] H. Brezis. *Analyse fonctionnelle.* Masson, 1987.

[3] G. Chen. Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain. *Journal de Mathématiques Pures et Appliquées*, 58:249–273, 1979.

[4] G. Chen, M. Delfour, A.M. Krall, and G. Payre. Modeling, stabilization and control of serially connected beams. *SIAM Journal on Control and Optimization*, 25(3):526–546, 1987.

[5] F. Conrad. Stabilization of beams by feedback point control. *SIAM Journal on Control and Optimization*, 28(2):423–437, 1990.

[6] F. Conrad, J. Leblond, and J.P. Marmorat. Energy decay estimates for a beam with nonlinear boundary feedback. In *Proceedings of the Workshop on Stabilization of Flexible Structures*, COMCON, 1989. Montpellier, to appear.

[7] F. Conrad, J. Leblond, and J.P. Marmorat. Stabilization of second order evolution equation by unbounded nonlinear feedback. In A. El Jai and M. Amouroux, editors, *Fifth Symposium on Control of Distibuted Parameter Systems*, IFAC, 1989. Perpignan.

[8] R.F. Curtain. Equivalence of input-output stability and exponential stability for infinite-dimensional systems. *Mathematical Systems Theory*, 21:19–48, 1988.

[9] R.F. Curtain and G. Weiss. Well posedness of triples of operators. 1989. To appear.

[10] A. Haraux. Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps. *Portugalica Mathematica*, 56(3):245–258, 1989.

[11] L.F. Ho and D.L. Russell. Admissible input elements for systems in Hilbert space and a Carleson measure criterion. *SIAM Journal on Control and Optimization*, 21:614–640, 1983.

[12] A.E. Ingham. Some trigonometrical inequalities with applications to the theory of series. *Mathematische Zeitschrift*, 41:367–379, 1936.

[13] I. Lasiecka and R. Triggiani. Riccati equations for hyperbolic partial differential equations with $L^2(0,T;L^2(\Gamma))$-Dirichlet boundary terms. *SIAM Journal on Control and Optimization*, 24(5):884–925, 1986.

[14] J. Leblond. *Stabilisation de Structures Flexibles*. Thèse de Doctorat, Université de Nice, Ecole des Mines de Paris, 1989.

[15] J. Leblond and J.P. Marmorat. Stabilization of a vibrating beam: a regularity result. In A.V. Balakrishnan and J.P. Zolesio, editors, *Proceedings of the Workshop on Stabilization of Flexible Structures*, pages 162–183, COMCON, Optimization Software Inc., 1987. Montpellier.

[16] J.L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications*. Volume 1-2, Dunod, 1968.

[17] A.J. Pritchard and D. Salamon. The linear quadratic control problem for infinite dimensional systems with unbounded input and output operators. *SIAM Journal on Control and Optimization*, 25:121–144, 1987.

[18] P. Rideau. *Contrôle d'un assemblage de poutres flexibles par des capteurs-actionneurs ponctuels: étude du spectre du système*. Thèse de Docteur-Ingénieur, Ecole des Mines de Paris, 1985.

[19] D.L. Russell. Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *SIAM Review*, 20:639–739, 1978.

[20] E. Zuazua. Exponential decay for semilinear wave equations with localized damping. *Comm. in P.D.E.*, 15(2):205-235, 1990.

[21] E. Zuazua. Sur l'optimalité des feedbacks de stabilisation. *C. R. Acad. Sci. Paris*, 309(1):547-552, 1989.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

ISSN 0249-6399